Learning Goals: Power Series

- Master the art of using the ratio test to find the radius of convergence of a power series.
- Learn how to find the series associated with the end points of the interval of convergence.
- Become familiar with the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- Become familiar with the limit $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!}$.
- Know the Power series representation of e^x
- Learn how to manipulate the power series representation of e^x using tools previously mastered such as substitution, integration and differentiation.
- Learn how to use power series to calculate limits.

Power Series, Stewart, Section 11.8

In this section, we will use the ratio test to determine the Radius of Convergence of a given power series. We will also determine the interval of convergence of the given power series if this is possible using the tests for convergence that we have already developed, namely, recognition as a geometric series, harmonic series or alternating harmonic series, telescoping series, the divergence test along with the fact that a series is absolutely convergent converges. The ratio and root tests will be inconclusive at the end points of the interval of convergence in the examples below since we use them to determine the radius of convergence. We may not be able to decide on convergence or divergence for some of the end points with our current tools. We will expand our methods for testing individual series for convergence in subsequent sections and tie up any loose ends as we proceed. We recall the definition of the Radius of Convergence and Interval of Convergence of a power series below.

Theorem For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities for the the values of x for which the series converges :

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Definition The **Radius of convergence (R.O.C.)** of the power series
is the number R in case 3 above.
In case 1, the Radius of convergence is 0 and
in case 2, the Radius of convergence is ∞.

We see that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges within some interval centered at a and diverges outside that interval. The **Interval of Convergence** of a power series is the interval that consists of all values of x for which the series converges.

- In case 1 above, the interval of convergence is a single point $\{a\}$.
- In case 2 above the interval of convergence is $(-\infty, \infty)$.
- In case 3 above the interval of convergence may be

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R].$$

Example The power series below is centered at 1.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$$

(a) Use the ratio test to determine the radius of convergence for this series.

(b) What are the values of x at the endpoints of the interval of convergence?

(c) Write down the two series that we must test for convergence/divergence to determine the interval of convergence. (We have not yet developed the tools to test these series for convergence/divergence. We will do so in a later lecture).

Example: VIP Series (a) Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

(b) **Very Impotant result for later** Show $\boxed{\lim_{n\to\infty} \frac{|x|^n}{n!} = 0}$ for all values of x.

(c) We will now show that te sum of this power series is in fact a very familiar function. Recall the binomial

Binomial theorem For any integer $m \geq 1$

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{3!}a^{m-3}b^3 + \cdots + b^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k}b^k.$$

The numbers $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$ $k \in \{0, 1, 2, \ldots k\}$ are called the binomial coefficients.

(i) Use L'Hospital's rule to show that for any given value of x

$$\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = e^x.$$

(ii) Verify that for any value of x by the binomial theorem;

$$\left(1+\frac{x}{m}\right)^m = 1 + x + \frac{m(m-1)}{2!}\frac{x^2}{m^2} + \frac{m(m-1)(m-2)}{3!}\frac{x^3}{m^3} + \cdots + \frac{m!}{m!}\frac{x^m}{m^m}.$$

(iii) As $m \to \infty$ the right hand side approaches a series of the form (this is not obvious but it is true)

$$1 + \sum_{k=1}^{\infty}\left(\lim_{m\to\infty}\frac{m(m-1)\ldots(m-(k-1))}{m^k}\right)\frac{x^k}{k!}$$

Show that for any given value of k the coefficient of $\frac{x^k}{k!}$ is

$$\lim_{m\to\infty}\frac{m(m-1)\ldots(m-(k-1))}{m^k} = 1.$$

(iv) Using your calculations above show that for any given value of x

$$e^x = \sum_{n=0}^{\infty}\frac{x^k}{k!},$$

by equating the limits (as m approaches infinity) of both sides of the equation in part (ii).

Example Consider the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1},$$

(a) Use the ratio test to determine the radius of convergence for this series.

(b) What are the values of x at the endpoints of the interval of convergence?

(c) Write down the two series that we must test for convergence/divergence to determine the interval of convergence. (We have not yet developed the tools to test these series for convergence/divergence. We will do so in a later lecture).

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{n(2x-1)^n}{5^n}$$

Let us update our table to include our power series representation for the exponential function.

function	Power series Repesentation	Interval
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\dfrac{1}{1+x^k}$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$	$-1 < x \leq 1$
$\arctan(x)$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \overset{?}{<} x \overset{?}{<} 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$

Applications (a) Use the method of substitution to give a power series representation for e^{x^2}.

(b) Use the method of integration to derive a power series representation for $\int e^{x^2}\,dx$.

(c) Use power series to find the imit

$$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2}{x^4}.$$